Section - 7

Homogeneous eqn of order n

Let $L[y] = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y^{(1)}$

Where $a_1, a_2 \dots a_n$ are Constant

Let us Solve $L[y] = 0$

Now,

$$L[e^{rx}] = r^n e^{rx} + a_1 r^{n-1} e^{rx} + a_2 r^{n-2} e^{rx} + \dots + a_n e^{rx}$$

$$= P(r) e^{rx} \rightarrow (1)$$

Where

$$P(r) = r^n + a_1 r^{n-1} + a_2 r^{n-2} + \dots + a_n$$

Here $P(r)$ is called the characteristic Polynomial of L.

Theorem: 11

Let $r_1, r_2 \dots r_s$ be the distinct roots of the characteri Polynomial P and suppose r_i has a multiplicity m_i (Thus $m_1 + m_2 + \dots + m_s = n$) Then the 'n' functions

$$e^{r_1 x}, x e^{r_1 x} \dots x^{m_1 - 1} e^{r_1 x}.$$

$$e^{r_2 x}, x e^{r_2 x} \dots x^{m_2 - 1} e^{r_2 x}$$

$$\vdots$$

$$e^{r_s x}, x e^{r_s x} \dots x^{m_s - 1} e^{r_s x}$$ are soln of $L[y] = 0$

Proof:

r_1 is a roots of multiplicity m_1 of P Then

$$P(r_1) = 0, \; P'(r_1) = 0 \dots P^{m_1 - 1}(r_1) = 0$$

Differentiable the eqn $L(e^{rx}) = P(r) e^{rx}$ k-times w.r.t 'r'

We get,

$$\frac{\delta^k L[e^{rx}]}{\delta r_k} = L\left[\frac{\delta k}{\delta r_k} e^{rx}\right]$$

$$= L[x^k e^{rx}]$$

$$= [P^k(r) + k P^{k-1}(r) + \frac{k(k-1)}{2!} P^{(k-2)}(r) x^2 + \dots + P(r) x^k] e^{rx}$$

If f, g are two functions having k derivatives

Then $(fg)^{k} = f^{(k)}g + k f^{(k-1)} g' + \frac{k(k-1)}{2!} f^{(k-2)} g'' + \dots + f g^{(k)}$

Thus for $k = 0, 1, \dots m-1$

we see that $x^{k}, e^{r_i x}$ is a soln of $L[y]=0$

Repeatly thus process for each root of p

We get the result.

Definition

The n functional $\varphi_1, \varphi_2 \dots \varphi_n$ on an interval I are said to be linearly dependent on I if there are constants $c_1, c_2 \dots c_n$ not all zero such that,

$c_1 \varphi_1(x) + c_2 \varphi_2(x) + \dots + c_n \varphi_n(x) = 0$ for all the functions $\varphi_1, \varphi_2 \dots \varphi_n$ are said to be linearly independent on I. If they are not linearly dependent on I.

Theorem: 12

Then n solns of $L[y]=0$, g_n is previous theorem are linearly independent on any interval I.

Proof:

Suppose we have 'n' constants c_{ij} $(i=1,2 \dots s)(j=0,1,\dots m_i)$ such that,

$$\sum_{i=1}^{s} \sum_{j=0}^{m_i-1} c_{ij}\, x^{j} e^{r_i x} = 0 \rightarrow (1)$$

Keeping i fixed and summing over j we take

$P_i(x) = \sum_{j=0}^{m_i-1} c_{ij} x^{j}$ be the polynomial coefficient of $e^{r_i x}$ in (1)

Thus we have

$$P_1(x) e^{r_1 x} + P_2(x) e^{r_2 x} + \dots + P_s(x) e^{r_s x} = 0 \rightarrow ② \quad \text{on } I.$$

Suppose that not all the constants c_{ij} are '0' then there will be atleast one of the polynomials P_i which is not identically zero on I.

The roots r_i if necessary we can assume that P_s is not identically zero on I

from ② we have.

$$P_1(x) + P_2(x) e^{(r_2 - r_1)x} + P_3(x) e^{(r_3 - r_1)x} + \dots + P_s(x) e^{(r_s - r_1)x} = 0 \rightarrow ③$$

Diff ③ sufficiently many times (atmost m_1 times) we can reduce $P_1(x)$ to 0 multiplying $e^{(r_s - r_1)x}$ remains unchanged as well as the non-identically vanishing character of any of there polynomial,

We obtain an expression of the form.

$$Q_2(x) e^{(r_2 - r_1)x} + Q_s(x) e^{(r_s - r_1)x} = 0 \rightarrow ④$$

(or)

$$Q_2(x) e^{r_2 x} + \dots + Q_s(x) e^{r_s(x)} = 0 \quad \text{on } I$$

where the Q_i are polynomial

$$\deg Q_i = \deg P_i$$

And Q_i does not vanish identically.

continuting this process we finally situdation where $R_s(x) e^{r_s(x)} = 0 \rightarrow ⑤$ on I

and R_s is a polyromials

$$\deg R_s = \deg P_s$$

which does not vanish identically on I.

But from ⑤ we get,

$$R_s(x) = 0 \quad \forall\ x \text{ in } I$$

This contradicts the assumption that

P_s is not identically zero on I.

Thus $P_s(x) = 0$ $\forall$ x in I and hence all that constants $c_{ij} = 0$ are linearly independent on the n solns on any interval I

Remark:

If $\varphi_1, \varphi_2 \dots \varphi_n$ are any m solns of $L[y]=0$ on an interval I and $c_1, c_2 \dots c_m$ are any m constants then.

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_m\varphi_m \text{ is also a soln.}$$

Since $L[\varphi] = c_1 L[\varphi_1] + \dots + c_m L[\varphi_m] = 0$

As in the $n=2$, every soln of $L[y]=0$ is a linear combination of n linearly independent soln

Defn:

The wronskian $W(\varphi_1, \varphi_2 \dots \varphi_n)$ of n functions $\varphi_1, \varphi_2 \dots \varphi_n$ having $(n-1)$ derivatives on an interval I is defined to be the determinant function.

$$W(\varphi_1, \varphi_2 \dots \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \\ \varphi_1' & \varphi_2' & \dots & \varphi_n' \\ \vdots & & & \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \dots & \varphi_n^{(n-1)} \end{vmatrix}$$

Its value at any x in I being $W(\varphi_1, \varphi_2 \dots \varphi_n)$

Theorem : 12 (a)

If $\phi_1, \phi_2 \ldots \phi_n$ are n soln of $L(y)=0$ on an interval I, They are linearly independent on I iff $W(\phi_1, \phi_2 \ldots \phi_n) \neq 0$, for all x in I

Proof

Let us suppose $W(\phi_1, \phi_2 \ldots \phi_n)(x) \neq 0$, $\forall x \in I$ and let $c_1, c_2 \ldots c_n$ be constants $\ni$

$c_1\phi_1(x) + c_2\phi_2(x) + \ldots + c_n\phi_n(x) = 0 \rightarrow (1) \quad \forall x \in I$

Also by differential (1) w.r.t. x

$c_1\phi_1'(x) + c_2\phi_2'(x) + \ldots + c_n\phi_n'(x) = 0 \rightarrow (2) \quad \forall x \in I$

for fixed x the eqn (1) and (2) are linearly homogeneous equations satisfied by $c_1, c_2 \ldots c_n$

The determinant of the coefficients is $W(\phi_1, \phi_2 \ldots \phi_n)(x)$ which is not zero.

$\therefore c_1 = 0, c_2 = 0, \ldots c_n = 0$ is only soln of (1) and (2)

$\therefore \phi_1, \phi_2 \ldots \phi_n$ are linearly independent on I

There exists

$W(\phi_1, \phi_2 \ldots \phi_n)(x_0) = 0$

Take the equations,

$c_1\phi_1(x_0) + c_2\phi_2(x_0) + \ldots + c_n\phi_n(x_0) = 0$

$c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \ldots + c_n\phi_n'(x_0) = 0$

where $c_1, c_2 \ldots c_n$ are constants

These are linearly homogeneous equations

Here the determinant of the constant

is $W(\phi_1, \phi_2 \ldots \phi_n)(x_0) = 0$

$\therefore$ Atleast one of the constants $c_1, c_2 \ldots c_n$ is not zero

for the constants $c_1, c_2 \ldots c_n$

We have,

$$c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x) = 0$$

one of $c_1, c_2 \dots c_n$ is not zero $\forall x \in I$

let $c_1, c_2 \dots c_n$ be a soln. consider the function

$$\psi = c_1\psi_1 + c_2\psi_2 + \dots + c_n\psi_n.$$

Now $L[\psi] = 0$

$$\psi'(x_0) = 0, \quad \psi(x_0) = 0$$

$\Rightarrow \phi_1, \phi_2 \dots \phi_n$ are linearly independent on I.

The contradicts the statement of the theorem.

$$\therefore W(\phi_1, \phi_2 \dots \phi_n)(x) \neq 0, \quad \forall x \in I$$

Hence proved

Section - 8

Initial value problems for n^{th} order equations.

consider the equation.

$$L[y] = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$$

an IVP for $L[y] = 0$ is a problem of finally a soln ϕ which satisfy the conditional

$$\phi(x_0) = \alpha_1, \ \phi'(x_0) = \alpha_2 \dots \phi^{n-1}(x_0) = \alpha_n.$$

where $\alpha_1, \alpha_2 \dots \alpha_n$ are constants x_0 is some real numbers

ie) initial point.

Note:

The soln of ϕ has prescribed values for it and its 1^{st} $(n-1)$ derivatives at some point x_0

It is denoted by

$$L[y] = y^n + a_1 y^{n-1} + \dots + a_n y = 0$$

$$y(x_0) = \alpha_1, \ y'(x_0) = \alpha_2 \dots y^{n-1}(x_0) = \alpha_n \ \text{on } I$$

Defn:

Let $\varphi(x)$ be any soln of $L[y]=0$ Then we define $\|\varphi(x)\|$ by,

$$\|\varphi(x)\| = [|\varphi(x)|^2 + |\varphi'(x)|^2 + \dots + |\varphi^{n-1}(x)|^2]^{1/2}$$

It is the length of L and the positive square root is taken

Theorem: 13

Let φ be any soln of

$L[y] = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$ on an interval I containing a point x_0, Then for all x in I

$$\|\varphi(x_0)\| e^{-K|x-x_0|} \le \|\varphi(x)\| \le \|\varphi(x_0)\| e^{K|x-x_0|}$$

Where $K = 1 + |a_1| + \dots + |a_n|$

Proof:

Let $u(x) = \|\varphi(x)\|^2$

$$= |\varphi(x)|^2 + |\varphi'(x)|^2 + \dots + |\varphi^{n-1}(x)|^2$$

$$\Rightarrow u = \varphi.\bar{\varphi} + \varphi'\bar{\varphi}' + \dots + \varphi^{(n-1)}\overline{\varphi^{(n-1)}}$$

Diff w.r.t x we get,

$$u' = \varphi'.\bar{\varphi} + \varphi.\bar{\varphi}' + \varphi''.\bar{\varphi}' + \varphi'.\bar{\varphi}'' + \dots + \varphi^{(n)}\overline{\varphi^{(n-1)}} + \varphi^{(n-1)}\overline{\varphi^{(n)}}$$

ie) $|u'| \le |\varphi'||\varphi| + |\varphi||\varphi'| + |\varphi''||\varphi'| + |\varphi'||\varphi''| + \dots + |\varphi^{(n)}||\varphi^{(n-1)}| + |\varphi^{(n-1)}||\varphi^{(n)}|$

$$|u'| \le 2|\varphi'||\varphi| + 2|\varphi'||\varphi''| + \dots + 2|\varphi^{(n-1)}||\varphi^{(n)}| \rightarrow (1)$$

Since φ Satisfies $L[y]=0$ we have

$$\varphi^{(n)} = -[a_1\varphi^{(n-1)} + a_2\varphi^{(n-2)} + \dots + a_n\varphi)$$

$$|\varphi^{(n)}| \le |a_1||\varphi^{(n-1)}| + |a_2||\varphi^{(n-2)}| + \dots + |a_n||\varphi| \rightarrow (2)$$

Sub (2) in (1) we get,

$|u'| \le 2|\varphi||\varphi'| + 2|\varphi'||\varphi''| + \dots + 2|\varphi^{(n-1)}|[|a_1||\varphi^{(n-1)}| + |a_2||\varphi^{(n-2)}| + \dots + |a_n||\varphi|]$

$\le 2|\varphi||\varphi'| + 2|\varphi'||\varphi''| + \dots + 2|a_1||\varphi^{(n-1)}|^2 + 2|a_2||\varphi^{(n-1)}||\varphi^{(n-2)}| + \dots$

using The inequality

$$2|b||c| \le |b|^2+|c|^2$$

we get,

$$|u'| \le [|\varphi|^2+|\varphi'|^2]+[|\varphi'|^2+|\varphi''|^2]+\cdots+2|a_1||\varphi^{(n-1)}|^2+2|a_2||\varphi^{(n-1)}||\varphi^{(n-2)}|$$

$$+\cdots+2|a_n||\varphi||\varphi^{(n-1)}|$$

$$=[|\varphi|^2+|\varphi'|^2]+[|\varphi'|^2+|\varphi''|^2]+\cdots+2|a_1||\varphi^{(n-1)}|^2+|a_2|[|\varphi^{(n-1)}|^2+|\varphi^{(n-2)}|^2]$$

$$+\cdots+|a_n|[|\varphi|^2+|\varphi^{(n-1)}|^2]$$

$$=(1+|a_n|)|\varphi|^2+(2+|a_{n-1}|)|\varphi'|^2+\cdots+(2+|a_2|)|\varphi^{(n-2)}|^2+$$

$$(1+2|a_1|+|a_2|+\cdots+|a_n|)|\varphi^{(n-1)}|^2$$

$$|u'| \le 2(1+|a_1|+|a_2|+\cdots+|a_n|)|\varphi|^2+2(1+|a_1|+|a_2|+\cdots+|a_n|)|\varphi'|^2$$

$$\cdots+2(1+|a_1|+|a_2|+\cdots+|a_n|)|\varphi^{(n-1)}|^2$$

$$\le 2K|\varphi|^2+2K|\varphi'|^2+\cdots+2K|\varphi^{(n-1)}|^2$$

$$\le 2K[|\varphi|^2+|\varphi'|^2+\cdots+|\varphi^{(n-1)}|^2]$$

$$|u'| \le 2Ku$$

$$\Rightarrow -2Ku \le u' \le 2Ku \rightarrow (3)$$

Consider The right inequality which can be written as,

$$u'-2Ku \le 0$$

$$\Rightarrow e^{-2Ku}(u'-2Ku) \le 0$$

$$\Rightarrow (e^{-2Ku}u)' \le 0$$

If $x > x_0$ we integrate form x_0 to x and get

$$e^{-2Kx}u(x)-e^{-2Kx_0}u(x_0) \le 0$$

(i.e) $e^{-2Ku}u(x) \le e^{-2Kx_0}u(x_0)$

i.e) $u(x) \le u(x_0)e^{2K(x-x_0)}$

$$\Rightarrow \|\varphi(x)\| \le \|\varphi(x_0)\|e^{K(x-x_0)}, \quad x > x_0$$

Similarly the left inequality

$$\|\varphi(x_0)\| e^{-k|x-x_0|} \le \|\varphi(x)\| \qquad \text{if } x \nless x_0$$

Considering ③ for $x < x_0$ together with integration from x to x_0 gives

$$\|\varphi(x_0)\| e^{k(x-x_0)} \le \|\varphi(x)\| \le \|\varphi(x_0)\| e^{-k(x-x_0)} \qquad \text{for } x < x_0$$

$\to$ ⑤

Combining ④ and ⑤ we get,

$$\|\varphi(x_0)\| e^{-k|x-x_0|} \le \|\varphi(x)\| \le \|\varphi(x_0)\| e^{k|x-x_0|}$$

Hence the Theorem.

Theorem: 14 Uniqueness Theorem

Let $\alpha_1, \alpha_2 \dots \alpha_n$ be any n constants and let x_0 be any real number on any interval I containing x_0. $\exists$ atmost one soln φ of $L[y]=0$ satisfying

$$\varphi(x_0)=\alpha_1,\ \varphi'(x_0)=\alpha_2 \dots \varphi^{(n-1)}(x_0)=\alpha_n.$$

Proof:

Suppose φ, ψ are two solns of $L[y]=0$ on I satisfying the above Theorem conditions at x_0 then

$\theta = \varphi - \psi$ satisfy $L[\theta] = 0$

and $\theta(x_0) = \theta'(x_0) = \dots = \theta^{(n-1)}(x_0) = 0$

Thus $\|\theta(x_0)\| = 0$

from the inequality of the above Theorem,

$$\|\theta(x_0)\| e^{-k|x-x_0|} \le \|\varphi(x)\| \le \|\theta(x_0)\| e^{k|x-x_0|}$$

we get $\|\theta(x_0)\| = 0 \quad \forall x \text{ in } I$

ie) $\theta(x) = 0 \quad \forall x \text{ in } I$

$\Rightarrow \varphi = \psi$

Theorem : 15 Existence Theorem

Let $\alpha_1, \alpha_2 \dots \alpha_n$ be any n constants and let x_0 be any real number, Then There exists a soln of φ of $L[y]=0$ on $-\infty < x < \infty$ satisfying

$$\varphi(x_0) = \alpha_1, \quad \varphi'(x_0) = \alpha_2 \dots \varphi^{(n-1)}(x_0) = \alpha_n \rightarrow (1)$$

Proof:

Let $\varphi_1, \varphi_2 \dots \varphi_n$ be any set of n linearly independent soln of $L[y]=0$ on $-\infty < x < \infty$

It will be such That $\exists$ unique constants $c_1, c_2 \dots c_n$ such That,

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n$$ is a Soln of $L[y]=0$

satisfying the condition given by (1)

There constants would have to satisfy

$$c_1\varphi_1(x_0) + c_2\varphi_2(x_0) + \dots + c_n\varphi_n(x_0) = \alpha_1$$

$$c_1\varphi_1'(x_0) + c_2\varphi_2'(x_0) + \dots + c_n\varphi_n'(x_0) = \alpha_2$$

$$\vdots$$

$$c_1\varphi_1^{(n-1)}(x_0) + c_2\varphi_2^{(n-1)}(x_0) + \dots + c_n\varphi_n^{(n-1)}(x_0) = \alpha_n$$

This is a system of n linear eqns for $c_1, c_2 \dots c_n$

The displacement of The coefficient is just

$W(\varphi_1, \varphi_2 \dots \varphi_n)(x_0)$ which is not zero.

$\therefore$ There is a unique set of constant $c_1, c_2 \dots c_n$ satisfying for this choice $c_1, c_2 \dots c_n$ The function

$$\varphi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n$$

Gives The desired soln.

Theorem: 16

Let $\varphi_1, \varphi_2 \dots \varphi_n$ be n linearly independent soln of $L[y]=0$ on an interval I. if $c_1, c_2 \dots c_n$ are any constants Then $\varphi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n$ is a soln and every soln may be represented in This form

Proof:

We know that

$$L[\varphi] = c_1 L[\varphi_1] + c_2 L[\varphi_2] + \dots + c_n L[\varphi_n] = 0$$

Let φ be any soln of $L[y]=0$ and let x_0 be in I

Suppose $\varphi(x_0) = \alpha_1,\ \varphi'(x_0) = \alpha_2 \dots \varphi^{(n-1)}(x_0) = \alpha_n$

By Existence Theorem $\exists$ unique constants $c_1, c_2 \dots c_n$ SuchThat,

$\psi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n$ is a soln of $L[y]=0$, on I,

Satisfying,

$$\psi(x_0) = \alpha_1,\ \psi'(x_0) = \alpha_2 \dots \psi^{(n-1)}(x_0) = \alpha_n$$

By uniqueness Theorem $\varphi = \psi$

Hence $\varphi = c_1\varphi_1 + c_2\varphi_2 + \dots + c_n\varphi_n$.

Theorem: 17

Let $\varphi_1, \varphi_2 \dots \varphi_n$ be n soln of $L[y]=0$ on an interval I containing x_0 Then

$$W(\varphi_1, \varphi_2 \dots \varphi_n)(x) = e^{-a_1(x-x_0)} W(\varphi_1, \varphi_2 \dots \varphi_n)(x_0).$$

Proof:

Section-9

The Non homogeneous eqn of order n

Let us consider the non homogeneous eqn of order n name

$L[y] = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b(x)$

where b is a continuous fun of x on interval I and $a_1, a_2, \ldots a_n$ are constant.

Theorem: 18

Let b be continuous on an interval I and let $\varphi_1, \varphi_2, \ldots \varphi_n$ be n linearly independent soln of $L[y]=0$ on I. Every soln ψ of $L[y] = b(x)$ can be written as $\psi = \psi_p + c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n$ where ψ_p is a particular soln of $L[y] = b$. A particular soln ψ_p is given by

$$\psi_p(x) = \sum_{k=1}^{n} \varphi_k(x) \int_{x_0}^{x} \frac{W_k(t)\, b(t)}{W(\varphi_1, \varphi_2 \ldots \varphi_n)(t)}\, dt$$

Proof:

If ψ_p is a particular soln of $L[y] = b(x)$ and ψ is another soln then

$$L[\psi - \psi_p] = L(\psi) - L(\psi_p) = 0$$

Thus $\psi - \psi_p$ is a soln of homogeneous eqn $L(y) = 0$

$\therefore \psi = \psi_p + c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n$

Where ψ_p is a particular soln of $L[y] = b(x)$

The function $\varphi_1, \varphi_2 \ldots \varphi_n$ are n L.I soln of $L[y]=0$ and $c_1, c_2 \ldots c_n$ are constant.

To find particular soln ψ_p:

We proceed as in the case $n=1$

ie) By variation of constant method

ie) We have to find n functions $u_1, u_2 \ldots u_n$ such that $\psi_p = u_1\varphi_1 + u_2\varphi_2 + \cdots + u_n\varphi_n$ is a soln.

Diff w.r.t x

$$\psi_p' = (u_1'\varphi_1 + u_2'\varphi_2 + \cdots + u_n'\varphi_n) + (u_1\varphi_1' + u_2\varphi_2' + \cdots + u_n\varphi_n')$$

We choose,

$$u_1'\varphi_1 + u_2'\varphi_2 + \cdots + u_n'\varphi_n = 0$$

Then $\psi_p' = u_1\varphi_1' + u_2\varphi_2' + \cdots + u_n\varphi_n'$

Now, $\psi_p'' = u_1\varphi_1'' + u_1'\varphi_1' + u_2\varphi_2'' + u_2'\varphi_2' + \cdots + u_n\varphi_n'' + u_n'\varphi_n'$

$$= (u_1\varphi_1'' + u_2\varphi_2'' + \cdots + u_n\varphi_n'') + (u_1'\varphi_1' + u_2'\varphi_2' + \cdots + u_n'\varphi_n')$$

choose $u_1'\varphi_1' + u_2'\varphi_2' + \cdots + u_n'\varphi_n' = 0$

Then $\psi_p'' = u_1\varphi_1'' + u_2\varphi_2'' + \cdots + u_n\varphi_n''$

Proceedding in this way we get $u_1', u_2', \ldots u_n'$ satisfy

$$\left.\begin{aligned} u_1'\varphi_1 + u_2'\varphi_2 + \cdots + u_n'\varphi_n &= 0 \\ u_1'\varphi_1' + u_2'\varphi_2' + \cdots + u_n'\varphi_n' &= 0 \\ &\vdots \\ u_1'\varphi_1^{(n-1)} + u_2'\varphi_2^{(n-1)} + \cdots + u_n'\varphi_n^{(n-1)} &= 0 \end{aligned}\right\} \rightarrow (1)$$

with $\psi_p = u_1\varphi_1 + u_2\varphi_2 + \cdots + u_n\varphi_n$

$$\psi_p' = u_1\varphi_1' + u_2\varphi_2' + \cdots + u_n\varphi_n'$$

$$\psi_p'' = u_1\varphi_1'' + u_2\varphi_2'' + \cdots + u_n\varphi_n''$$

$$\vdots$$

$$\psi_p^{(n-1)} = u_1\varphi_1^{(n-1)} + u_2\varphi_2^{(n-1)} + \cdots + u_n\varphi_n^{(n-1)}$$

and $\psi_p^n = u_1\varphi_1^{(n)} + u_2\varphi_2^{(n)} + \cdots + u_n\varphi^{(n)}$

Hence $L(\psi_p) = u_1 L(\varphi_1) + u_2 L(\varphi_2) + \cdots + u_n L(\varphi_n) = b$

i.e) $L(\psi_p) = b$

[$\because$ $\varphi_1, \varphi_2, \ldots \varphi_n$ are solns of $L(y) = 0$]

and ψ_p is the soln of $L[\varphi] = b(x)$

The whole problem now reduces as to solve the linear system (1) for $u_1', u_2', \dots u_n'$

The determinant of the coefficient is $W(\varphi_1, \varphi_2, \dots \varphi_n)$ which is never zero, when $\varphi_1, \varphi_2 \dots \varphi_n$ are linearly independent solns of $L(y) = 0$

Hence there exists unique function satisfy (1) the solns are given by

$$u_k'(x) = \frac{W_k(x)\, b(x)}{W(\varphi_1, \varphi_2 \dots \varphi_n)(x)} \qquad k = 1, 2 \dots n$$

Where, W_k is the determinant obtained from $W(\varphi_1, \varphi_2 \dots \varphi_n)$ by replacing k^{th} colum

(i.e) $\varphi_k, \varphi_k' \dots \varphi_k^{(n-1)}$ by $(0, 0 \dots 0, 1)$

If x_0 is any point in I. we can take for u_k the function given by

$$u_k(x) = \int_{x_0}^{x} \frac{W_k(t)\, b(t)}{W(\varphi_1, \varphi_2 \dots \varphi_n)(t)}\, dt \qquad k = 1, 2 \dots n$$

The particular soln ψ_p becomes

$$\psi_p(x) = \sum_{k=1}^{n} \varphi_k(x) \int_{x_0}^{x} \frac{W_k(t)\, b(t)}{W(\varphi_1, \varphi_2 \dots \varphi_n)(t)}\, dt)$$

Hence the theorem

Problems

7. a) Find the solns of the following eqns $y'' + y = 0$

Soln

Given eqn is $y'' + y = 0$

The characteristic polynomial is

$$r^2 + 1 = 0$$

$$r = \pm i$$

The soln φ has a form

$$\varphi(x) = c_1 \cos x + c_2 \sin x$$

7.b) $y^{4}-y=0$

Soln:

Given eqn is $y^{4}-y=0$

The characteristic polynomial is

$r^4-1=0$

$r^4-1^4=0$

$(r^2+1^2)(r^2-1^2)=0$

$r^2+1^2=0, \quad r^2-1^2=0$

$r^2=-1, \quad r^2=1$

$r=\pm i \quad\quad r=\pm 1$

The soln φ has the form.

$\varphi(x)=c_1e^{x}+c_2e^{-x}+e^{0x}(c_3\cos x+c_4\sin x)$

$\varphi(x)=c_1e^{x}+c_2e^{-x}+c_3\cos x+c_4\sin x$

7.c) $y^{4}-5y''+4y=0$

Soln:

The characteristic polynomial is

$r^4-5r^2+4=0$

$r=-1, \quad r^3-r^2-4r+4=0$

$r=1, \quad r^2-4=0$

$r^2=4$

$r=\pm 2$

$\therefore$ The roots are $r=-1,1,2,-2$

$\therefore$ The soln is

$\varphi(x)=c_1e^{x}+c_2e^{-x}+c_3e^{2x}+c_4e^{-2x}$

-1	1	0	-5	0	4
	0	-1	1	4	-4
	1	-1	-4	4	L0

1	1	-1	-4	4
	0	1	0	-4
	1	0	-4	L0

2	1	-6	12	-8
	0	2	-8	8
	1	-4	4	L0

$r^2-4r+4=0$

7.d) $y'''-8y=0$

$a^3-b^3=a^3-3a^2b+3ab^2-b^3$

Soln:

The characteristic polynomial is

$r^3-8=0$

$r^3-2^3=0$

~~$(r-2)(r^2+2r+4)=0$~~

$\rightarrow r^3-6r^2+12r-8=0$

2	1	-6	12	-8
	0	2	-8	8
	1	-4	4	L0

$r=2, \quad r^2-4r+4=0$

$(r-2)(r-2)=0$

$r=2 \quad\quad r=2,2$

$\rightarrow$ The soln is $\varphi(x)=(c_1+c_2x+c_3x^2)e^{2x}$

$r-2=0, \qquad r^2+2r+4=0$

$r=2 \qquad r=-1\pm i\sqrt{3}$

$\therefore$ The soln ϕ has the form.

$\phi(x) = c_1 e^{2x} + e^{-x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x)$

$= c_1 e^{2x} + c_2 e^{-x}\cos\sqrt{3}x + c_3 e^{-x}\sin\sqrt{3}x$

(i)

7.e) $y^4 - 16y = 0$ (ii) $y^{(4)}+16y=0$.

Soln:

The characteristic polynomial is

$r^4-16=0$

$r^4-2^4=0$

$(r^2+2^2)(r^2-2^2)=0$

$r^2+2^2=0, \qquad r^2-2^2=0$

$r^2=-4 \qquad r^2=4$

$r=\pm i\sqrt{2} \qquad r=\pm 2$

The soln of $\phi(x)$ has the form.

$\phi(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3\cos\sqrt{2}x + c_4\sin\sqrt{2}x$

7.f) $y''' - 5y'' + 6y' = 0$

Soln:

The characteristic polynomial

$r^3-5r^2+6r=0$

$r(r^2-5r+6)=0$

$r=0, \qquad r^2-5r+6=0$

$(r-2)(r-3)=0$

$r-2=0, \ r-3=0$

$r=2,3$

The roots are $r=0,2,3$

The soln of $\phi(x)$ has the form.

$\phi(x) = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{3x}$

$\phi(x) = c_1 + c_2 e^{2x} + c_3 e^{3x}$

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7.g) $y''' - iy'' + 4y' - 4iy = 0$

Soln:

The characteristic polynomial is

$r^3 - ir^2 + 4r - 4i = 0$

$$\begin{array}{c|cccc} i & 1 & -i & 4 & -4i \\ & 0 & i & 0 & 4i \\ \hline 2i & 1 & 0 & 4 & \underline{10} \\ & 0 & 2i & -4 & \\ \hline & 1 & 2i & \underline{10} & \end{array}$$

The roots are

$r = i, 2i, -2i$

The soln of $\varphi(x)$ has the form

$\varphi(x) = c_1 e^{ix} + c_2 e^{2ix} + c_3 e^{-2ix}$

7.h) $y^{(4)} + 5y'' + 4y = 0$

Soln:

The characteristic polynomial is

$r^4 + 5r^2 + 4 = 0$

$(r^2+1)(r^2+4) = 0$

$r^2 + 1 = 0, \quad r^2 + 4 = 0$

$r = \pm i, \quad r = \pm 2i$

$\therefore$ The soln of $\varphi(x)$ has the form.

$\varphi(x) = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_3 \sin 2x$

(or)

$\varphi(x) = c_1 e^{ix} + c_2 e^{-ix} + c_3 e^{2ix} + c_4 e^{-2ix}$

7.i) a) $y''' - 3y' - 2y = 0$ b) $y''' - 3y' + 2y = 0$.

Soln:

a) The c.p is $r^3 - 3r - 2 = 0$

$$\begin{array}{c|cccc} 2 & 1 & 0 & -3 & -2 \\ & 0 & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & \underline{10} \end{array}$$

$r = 2, \quad r^2 + 2r + 1 = 0$

$(r+1)(r+1) = 0$

$r = -1, -1$

$\therefore$ The roots are

$r = 2, -1, -1$

The soln $\varphi(x)$ has the form.

$\varphi(x) = (c_1 + c_2 x) e^{-x} + c_3 e^{2x}$

b) $\phi(x) = (c_1 + c_2 x) e^{x} + c_3 e^{-2x}$.